Estimation of group action with energy constraint

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- Estimation of group action
  $\mathbb{R}$, **U(1), SU(2), and SO(3)**
  with average energy restriction
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- Application to uncertainty relation (Robertson type)

Estimation of group action

Given a projective unitary representation f of G on $\mathcal{H}$.

Input state	Unknown Unitary to be estimated	measurement	estimate
ρ	$f(g)$	M	$\hat{g}$

$$\rho \longrightarrow \boxed{f(g)} \longrightarrow M \longrightarrow \hat{g}$$

$\mathcal{E} = (\rho, M)$:Our operation

$R(g,\hat{g}) = R(e, g^{-1}\hat{g}) = R(e, \hat{g}g^{-1})$:error function

Average error when the true parameter is g

$$\mathcal{D}_{R,g}(\mathcal{E}) := \int_G R(g,\hat{g}) \mathrm{Tr} M(d\hat{g}) f(g)\rho f(g)^*$$

Bayesian: $\mathcal{D}_{R,\nu}(M) := \int_G \mathcal{D}_{R,g}(M)\nu(dg)$ prior: ν

Mini-max: $\mathcal{D}_R(M) := \max_g \mathcal{D}_{R,g}(M)$

Group covariant measurement

$\mathcal{H}$:Hilbert space

Holevo 1979

G :group

f :projective unitary representation

A POVM M taking values in G is called covariant if

$$f(g)M(B)f(g)^* = M(gB)$$

$\mathcal{M}(G)$:Set of POVMs taking the values in G

$\mathcal{M}_{\mathrm{cov}}(G)$:Set of covariant POVMs taking values in G

$M \in \mathcal{M}(G)$ is included in $\mathcal{M}_{\mathrm{cov}}(G)$

$$\Longleftrightarrow M(B) = M_T(B) := \int_B f(g) T f(g)^* \mu(dg)$$

Group-action-version of quantum Hunt-Stein theorem

Invariant probability measure μ exists for G when G is compact. Then, the following equations hold.

$$\min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) = \min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_{R,\mu}(\rho, M)$$

$$= \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(G)} \mathcal{D}_R(\rho, M)$$

$$= \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(G)} \mathcal{D}_{R,\mu}(\rho, M)$$

The following relation holds even when G is not compact.

$$\min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) = \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(G)} \mathcal{D}_R(\rho, M)$$

Fourier transform and inverse Fourier transform on group

$\hat{G}$: Set of irreducible unitary representation of G

$$L^2(\hat{G}) := \bigoplus_{\lambda \in \hat{G}} L^2(\mathcal{V}_\lambda)$$

$L^2(\mathcal{V}_\lambda)$: Set of HS operators on $\mathcal{V}_\lambda$

$\mathcal{F} : L^2(G) \to L^2(\hat{G})$: Fourier transform

$$(\mathcal{F}[\phi])_\lambda := \sqrt{d_\lambda} \int_G f_\lambda(g)^* \phi(g) \mu(dg)$$

$\mathcal{F}^{-1} : L^2(\hat{G}) \to L^2(G)$: Inverse Fourier transform

$$\mathcal{F}^{-1}[A](g) := \sum_{\lambda \in \hat{G}} \sqrt{d_\lambda} \mathrm{Tr} f_\lambda(g) A_\lambda$$

Optimization with Energy constraint via inverse Fourier transform

Energy constraint

$$\mathrm{Tr}\rho H \leq E$$

$$D_R(X) := \int_G R(e,\hat{g}) |\mathcal{F}^{-1}[X](\hat{g}^{-1})|^2 \mu(d\hat{g})$$

Our target is

$$\min_{\substack{\rho, M \in \mathcal{M}(G) \\ \mathrm{Tr}\rho H \leq E}} \mathcal{D}_R(\rho, M) = \min_{\substack{\rho:\mathrm{pure}, M \in \mathcal{M}_{\mathrm{cov}}(G), \\ \mathrm{Tr}\rho H \leq E}} \mathcal{D}_R(\rho, M)$$

$$= \min_{X \in L^2_{H,E}(\hat{G}), \|X\|^2 = 1} D_R(X)$$

$$L^2_{H,E}(\hat{G}) := \{ X \in L^2(\hat{G}) | \langle X | H | X \rangle \leq E \}$$

Example: $G=\mathbb{R} \quad (\hat{G}=\mathbb{R})$

$$R(g,\hat{g})=(g-\hat{g})^2, \qquad f(g)|\lambda\rangle = e^{ig\lambda}|\lambda\rangle, \qquad H=Q^2$$

$$\min_{\rho\in\mathcal{S}(L^2(\mathbb{R}))}\ \min_{M\in\mathcal{M}_{\mathrm{cov}}(\mathbb{R})}\left\{D_R(\rho,M)\,|\,\mathrm{Tr}\rho Q^2\le E\right\}$$

$$=\min_{|\phi\rangle\in L^2(\mathbb{R})}\left\{\int_{\mathbb{R}}\hat{g}^2\,|\mathcal{F}^{-1}[\phi](\hat{g})|^2\frac{d\hat{g}}{\sqrt{2\pi}}\,\middle|\,\int_{\mathbb{R}}\lambda^2\,|\phi(\lambda)|^2\frac{d\lambda}{\sqrt{2\pi}}\le E\right\}$$

$$=\min_{|\phi\rangle\in L^2(\mathbb{R})}\left\{\langle\phi|Q^2|\phi\rangle\,\middle|\,\langle\phi|P^2|\phi\rangle\le E\right\}=\frac{1}{4E}$$

Minimum is attained with $\varphi(\lambda)=e^{-\frac{\lambda^2}{4E^2}}\Big/\sqrt{E}$

Mathieu Function

Periodic differential operator

$$P^2 + 2q\cos 2Q$$

Minimum eigenvalue	Eigen function	space
$a_0(q)$	$\mathrm{ce}_0(\theta, q)$	$L^2_{\mathrm{p,even}}((-\frac{\pi}{2}, \frac{\pi}{2}])$
$b_2(q)$	$\mathrm{se}_2(\theta, q)$	$L^2_{\mathrm{p,odd}}((-\frac{\pi}{2}, \frac{\pi}{2}])$
$a_1(q)$	$\mathrm{ce}_1(\theta, q)$	$L^2_{\mathrm{a,even}}((-\frac{\pi}{2}, \frac{\pi}{2}])$
$b_1(q)$	$\mathrm{se}_1(\theta, q)$	$L^2_{\mathrm{a,odd}}((-\frac{\pi}{2}, \frac{\pi}{2}])$

Estimation of U(1)

$$R(g,\hat{g}) = 1 - \cos(g - \hat{g}), \qquad f(g)|k\rangle = e^{ikg}|k\rangle,$$

$$H = \sum_k k^2 |k\rangle\langle k|$$

$$\min_{\rho \in \mathcal{S}(L^2(\hat{\mathrm{U}}(1)))} \min_{M \in \mathcal{M}_{\mathrm{cov}}(\mathrm{U}(1))} \left\{ D_R(\rho, M) \,\middle|\, \mathrm{Tr}\rho H \le E \right\}$$

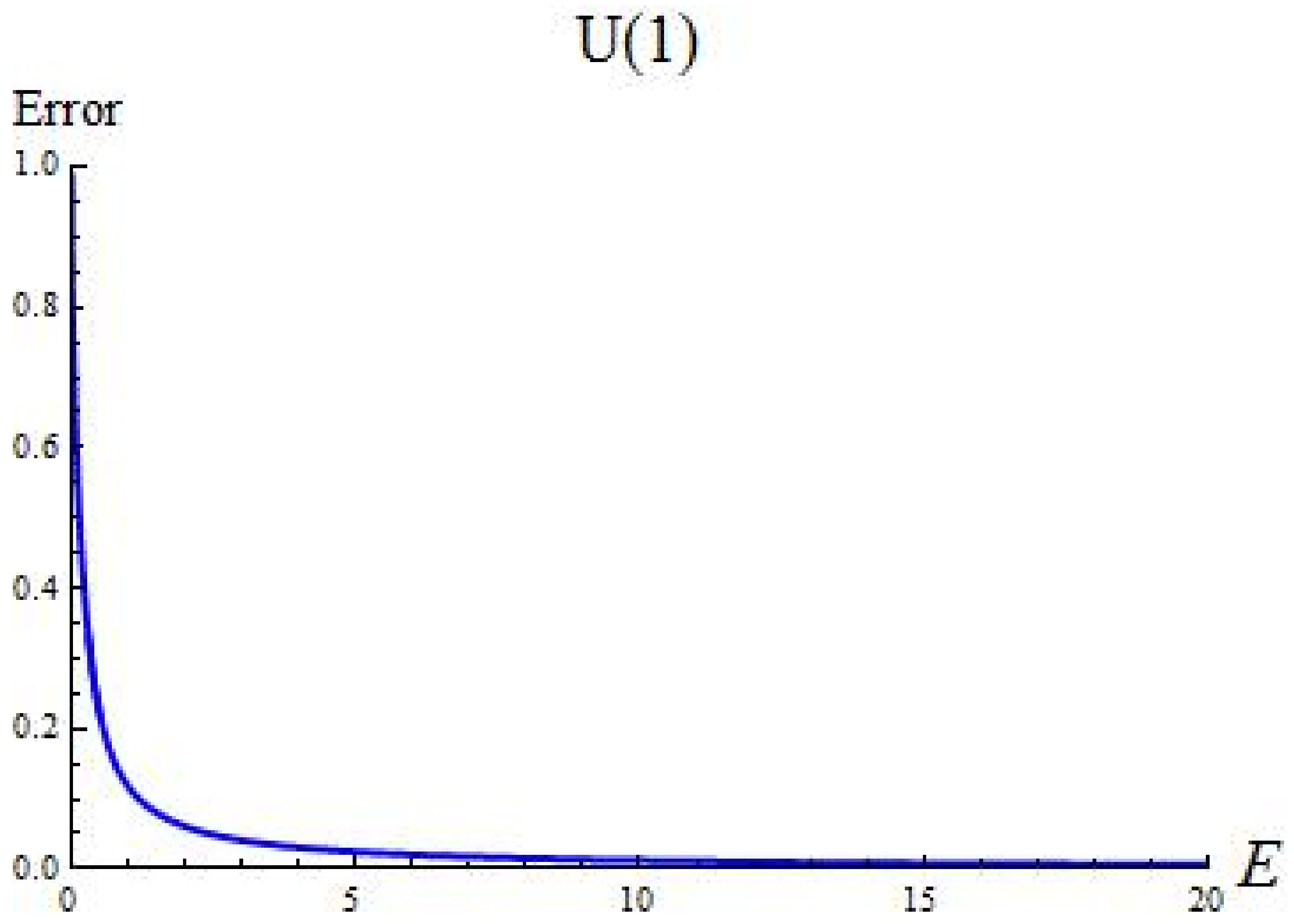
$$= \min_{|\phi\rangle \in L^2_{\mathrm{p,even}}((-\pi,\pi])} \left\{ \langle \phi | I - \cos Q | \phi \rangle \,\middle|\, \langle \phi | P^2 | \phi \rangle \le E \right\}$$

$$= \max_{s>0} \frac{s a_0(2/s)}{4} + 1 - sE$$

Optimal input is constructed by $\mathrm{ce}_0(\theta, q)$

$$\cong \begin{cases} \dfrac{1}{8E} - \dfrac{1}{128E^2} & \text{as } E \to \infty \\[2ex] 1 - \sqrt{2E} + \dfrac{7\sqrt{2}E^{\frac{3}{2}}}{16} & \text{as } E \to 0 \end{cases}$$

Graphs



Estimation of SU(2)

$$R(g,\hat{g}) = 1 - \frac{1}{2}\chi_{\frac{1}{2}}(\hat{g}g^{-1}), \qquad H = \bigoplus_{k=0}^{\infty} \frac{k}{2}\left(\frac{k}{2}+1\right) I_{\frac{k}{2}}$$

Reduce $L^2(\mathrm{S\hat{U}}(2))$ to $L^2_{\mathrm{p,odd}}((-\pi,\pi])$

$$\min_{\rho \in \mathcal{S}(L^2(\mathrm{S\hat{U}}(2)))} \min_{M \in \mathcal{M}_{\mathrm{cov}}(\mathrm{SU}(2))} \left\{ D_R(\rho, M) \,|\, \mathrm{Tr}\rho H \le E \right\}$$

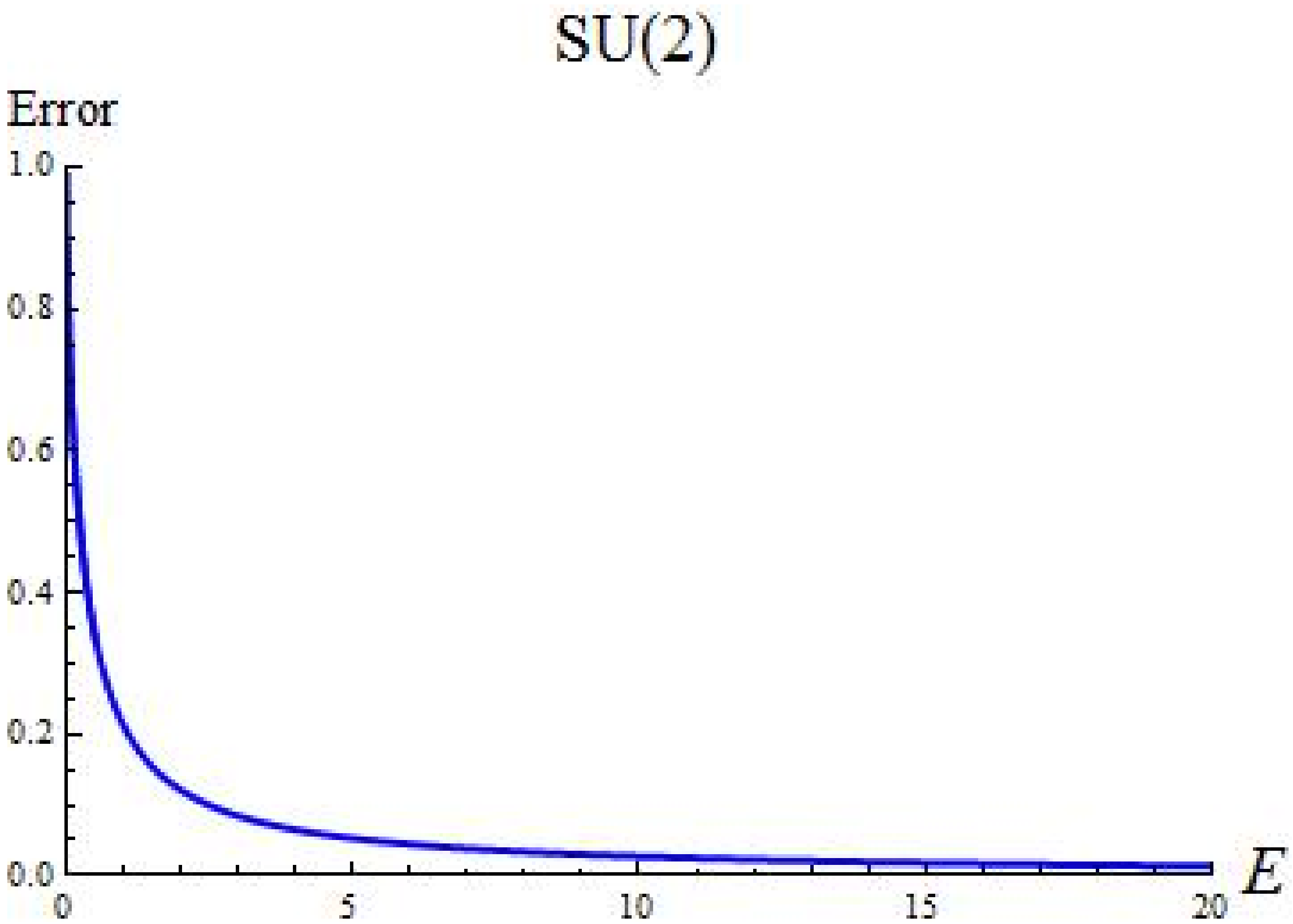
$$= \min_{|\phi\rangle \in L^2_{\mathrm{p,odd}}((-\pi,\pi])} \left\{ \left\langle \phi \right| I - \cos\frac{Q}{2} \left| \phi \right\rangle \middle| \left\langle \phi \right| P^2 \left| \phi \right\rangle \le E + \frac{1}{4} \right\}$$

$$= \max_{s>0} \frac{s b_2(8/s)}{4} + 1 - s\left(E + \frac{1}{4}\right)$$

$$\cong \begin{cases} \dfrac{9}{32E} - \dfrac{7 \bullet 3^3}{2^{11} E^2} & \text{as } E \to \infty \\[2ex] 1 - \dfrac{2}{\sqrt{3}}\sqrt{E} + \dfrac{5E^{\frac{3}{2}}}{6\sqrt{3}} & \text{as } E \to 0 \end{cases}$$

Optimal input is constructed by $\mathrm{se}_2(\theta, q)$

Graphs



Factor system of projective unitary representation

Chiribella 2011

Factor system

$$e^{i\theta(g,g')} := f(g) f(g') f(gg')^{-1}$$

$$\mathcal{L} := \{e^{i\theta(g,g')}\}_{g,g'}$$

$\hat{G}[\mathcal{L}]$: Set of projective irreducible representation with the factor system $\mathcal{L}$

$$D_R(X) := \int_G R(e,\hat{g}) |\mathcal{F}_{\mathcal{L}}^{-1}[X](\hat{g}^{-1})|^2 \mu(d\hat{g})$$

Estimation of SO(3)

$$R(g,\hat{g}) = \frac{1}{2}(3-\chi_1(\hat{g}g^{-1})), \quad H = \bigoplus_{k=0}^{\infty} \frac{k}{2}\left(\frac{k}{2}+1\right) I_{\frac{k}{2}}$$

Reduce $L^2(\mathrm{S\hat{O}}(3))$ to $L^2_{\mathrm{a,odd}}((-\pi,\pi])$ or $L^2_{\mathrm{p,odd}}((-\pi,\pi])$

$$\min_{\rho\in\mathcal{S}(L^2(\mathrm{S\hat{O}}(3)))} \min_{M\in\mathcal{M}_{\mathrm{cov}}(\mathrm{SO}(3))} \left\{ D_R(\rho,M) \mid \mathrm{Tr}\rho H \le E \right\}$$

$$= \begin{cases} \min_{\phi\in L^2_{\mathrm{a,odd}}} \{ \langle \phi | I - \cos Q | \phi \rangle \mid \langle \phi | P^2 | \phi \rangle \le E + \frac{1}{4} \} & \text{Integer case} \\ \min_{\phi\in L^2_{\mathrm{p,odd}}} \{ \langle \phi | I - \cos Q | \phi \rangle \mid \langle \phi | P^2 | \phi \rangle \le E + \frac{1}{4} \} & \text{Half integer case} \end{cases}$$

Integer case

$$\min_{\phi \in L^2_{\mathrm{a,odd}}} \{ \langle \phi | I - \cos Q | \phi \rangle \mid \langle \phi | P^2 | \phi \rangle \le E + \frac{1}{4} \}$$

$$= \max_{s>0} \frac{s a_1(2/s)}{4} + 1 - s(E + \frac{1}{4})$$

$$= \begin{cases} \frac{9}{8E} - \frac{81}{128 E^2} & E \to \infty \\ \frac{3}{2} - \frac{\sqrt{E}}{\sqrt{2}} - \frac{E}{4} & E \to 0 \end{cases}$$

Optimal input is constructed by $\mathrm{ce}_1(\theta, q)$

Half integer case

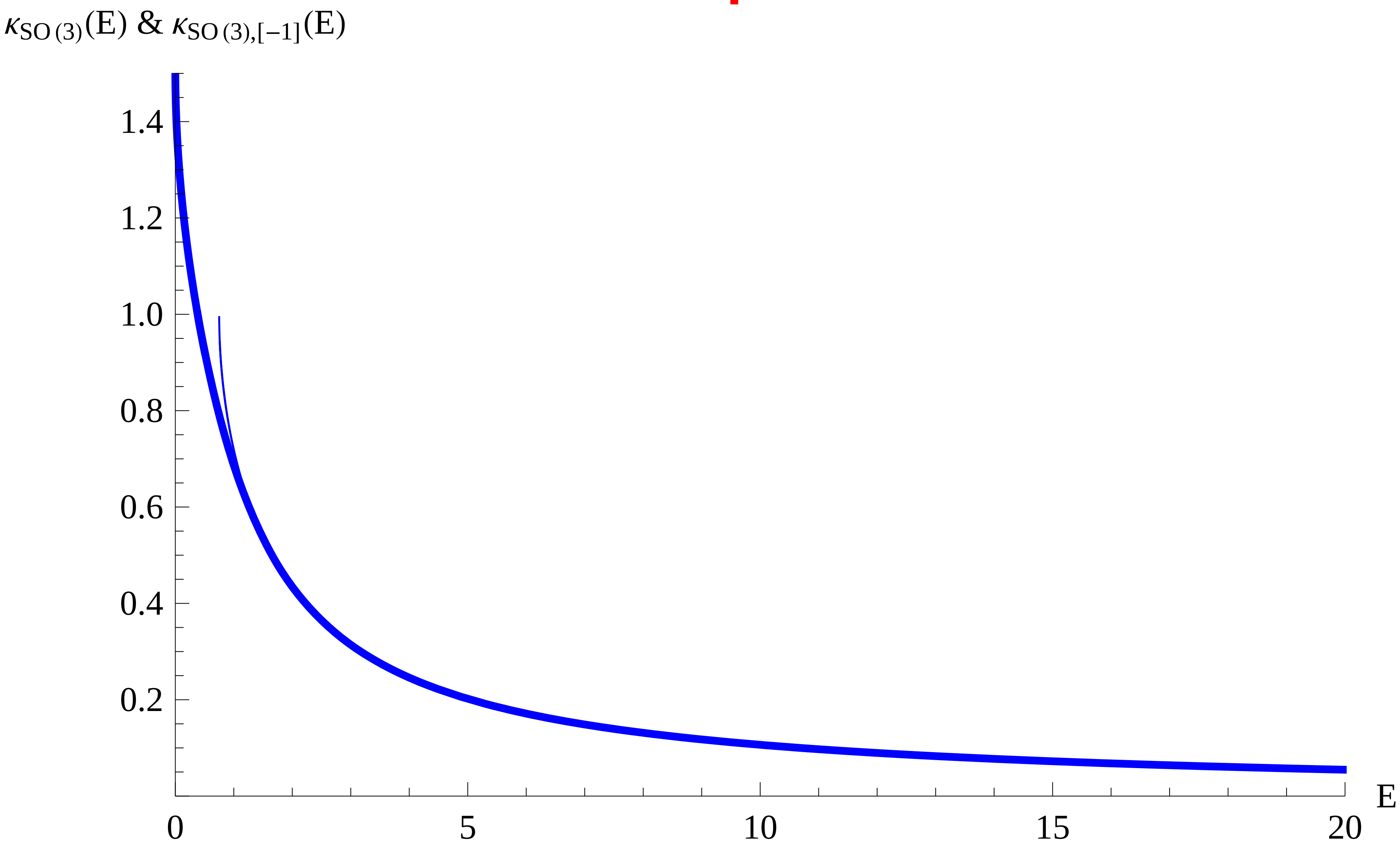
$$\min_{\phi\in L^2_{\mathrm{p,odd}}}\{\langle\phi|I-\cos Q|\phi\rangle \,|\, \langle\phi|P^2|\phi\rangle \le E+\frac{1}{4}\}$$

$$=\max_{s>0}\frac{sb_2(2/s)}{4}+1-s(E+\frac{1}{4})$$

$$=\begin{cases}\dfrac{9}{8E}-\dfrac{81}{128E^2} & E\to\infty\\[2ex] 1-\dfrac{1}{\sqrt{3}}(E-\dfrac{3}{4})^{\frac{1}{2}}+\dfrac{5}{48\sqrt{3}}(E-\dfrac{3}{4})^{\frac{3}{2}} & E\to\dfrac{3}{4}\end{cases}$$

Optimal input is constructed by $\mathrm{se}_2(\theta,q)$

Graphs



Thick line expresses the projective case,
and Normal line expresses the representation case

Non-compact Example: $G = \mathbb{R}^2$

f : Heisenberg representation

$$|X\rangle \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$$

multiplicity

Minimize

$$\int_{\mathbb{R}^2} (x^2 + y^2) |\mathcal{F}^{-1}[X](\frac{x+yi}{\sqrt{2}})|^2 \, dxdy$$

under

$$\langle X | (Q^2 + P^2) \otimes I | X \rangle \leq E$$

Minimum value: $\dfrac{1}{2E}$

How to derive minimum

Fourier transform $\mathcal{F} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$

$$\mathcal{F}^{-1}(Q \otimes I)\mathcal{F} = P_2 - \frac{1}{2}Q_1, \quad \mathcal{F}^{-1}(P \otimes I)\mathcal{F} = -P_1 - \frac{1}{2}Q_2$$

Via $\phi = \mathcal{F}^{-1}[X]$, minimizing problem is equivalent with

Minimize $\langle \phi | Q_1^2 + Q_2^2 | \phi \rangle$

under $\langle \phi | (P_2 - \frac{1}{2}Q_1)^2 + (-P_1 - \frac{1}{2}Q_2)^2 | \phi \rangle \leq E$

By choosing suitable coordinate, minimizing problem is equivalent with

Minimize $\langle \phi | Q_1^2 + Q_2^2 | \phi \rangle$

under $\langle \phi | P_1^2 + P_1^2 | \phi \rangle \leq E$

Uncertainty relation

Minimum value

$1/2E$

Practical realization of asymptotically optimal estimator

$$G = \mathrm{U}(1) \quad \mathcal{H} = \langle |k\rangle \rangle, \quad H = \sum_k k^2 |k\rangle\langle k|$$

Assume that ϕ satisfies $\sum_k k\,|\langle k|\phi\rangle|^2 = 0$

$\mathcal{F}[\phi]$ is even function

$$|\phi\rangle \to U \to \boxed{M} \to \theta_1$$

$$|\phi\rangle \to U \to \boxed{M} \to \theta_2$$

$$\vdots$$

$$|\phi\rangle \to U \to \boxed{M} \to \theta_n$$

MLE: $\theta_1, \theta_2, \ldots, \theta_n \to \hat{\theta}^n$

This method attains the optimal performance.

Practical realization of asymptotically optimal estimator

$$G = \mathrm{SU}(2) \qquad \mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_{\lambda}$$

Assume that the support of ϕ contains both of integer rep. and half integer rep.

MLE

$$|\phi\rangle \to U \to \boxed{M} \to \theta_1$$
$$|\phi\rangle \to U \to \boxed{M} \to \theta_2$$
$$\vdots$$
$$|\phi\rangle \to U \to \boxed{M} \to \theta_n$$
$$\Rightarrow \quad \theta_1, \theta_2, \ldots, \theta_n \to \hat{\theta}^n$$

This method attains the optimal performance.

Practical realization of asymptotically optimal estimator

$G = \mathbf{SO(3)}$ $\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_{\lambda}$

Assume that the support of ϕ contains only integer rep. or half integer rep.

MLE

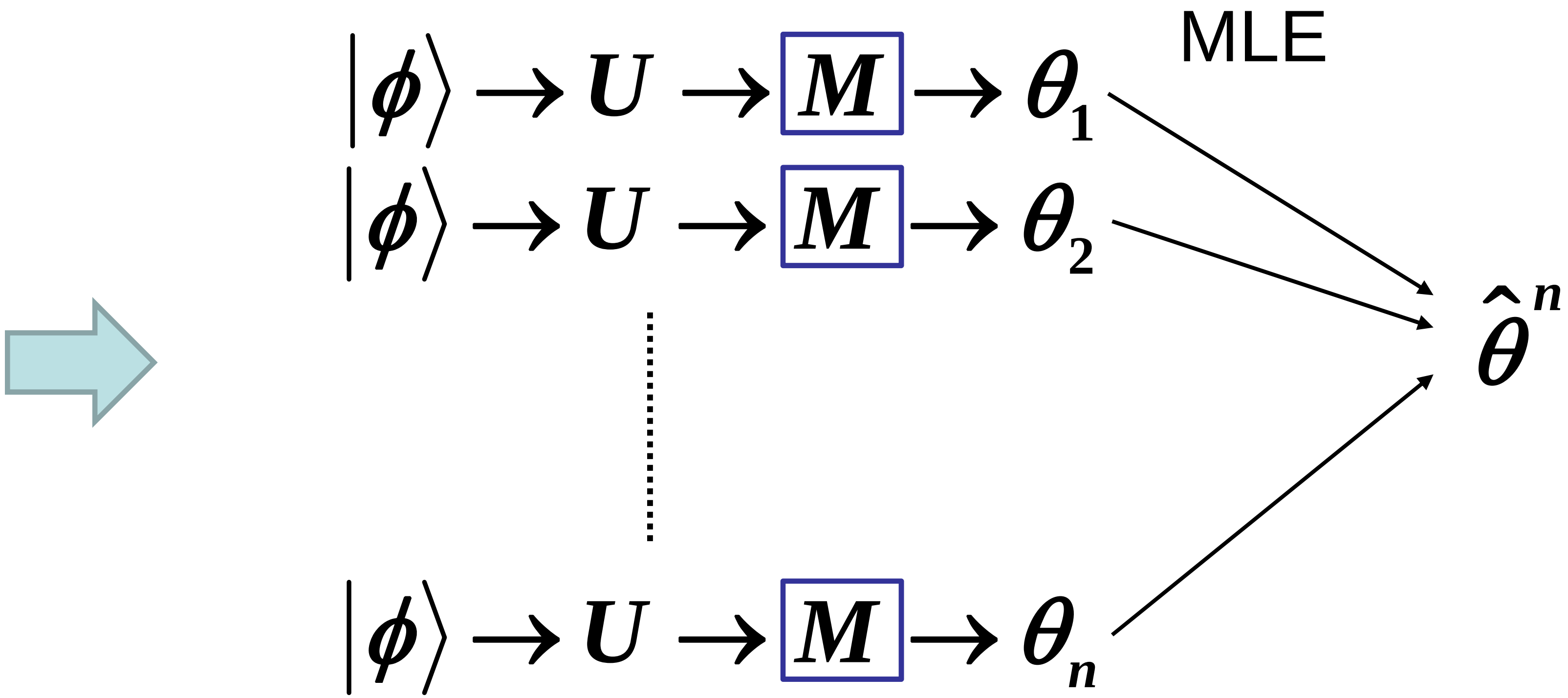
$$|\phi\rangle \to U \to \boxed{M} \to \theta_1$$
$$|\phi\rangle \to U \to \boxed{M} \to \theta_2$$
$$\vdots$$
$$|\phi\rangle \to U \to \boxed{M} \to \theta_n$$

$$\Rightarrow \hat{\theta}^n$$

This method attains the optimal performance.

Implication of these optimal estimators

When we consider the energy constraint,
entangled input state and measurement with entangled basis do not enhance the quality of estimation.



This method attains the optimal performance.

Uncertainty relation on $L^2_p((-\pi,\pi])$

$$= L^2(\mathrm{U}(1)) = L^2(S^1)$$

$$\Delta^2_\phi(\cos Q, \sin Q) := \Delta^2_\phi \cos Q + \Delta^2_\phi \sin Q$$

$$\min_{\phi \in L^2_p((-\pi,\pi])} \left\{ \Delta^2_\phi(\cos Q, \sin Q) \,|\, \Delta^2_\phi P \le E \right\}$$

$$= \max_{s>0} 1 - (sE - \frac{sa_0(2/s)}{4})^2$$

The minimum is realized by $\mathrm{ce}_0(\frac{\theta}{2}, -\frac{2}{s_E})$

$$s_E := \arg\max_{s>0} 1 - (sE - \frac{sa_0(\frac{2}{s})}{4})^2$$

Uncertainty relation on $L^2(\mathrm{SU}(2)) = L^2(S^3)$

$$g \mapsto (x_0(g), x_1(g), x_2(g), x_3(g)) \in S^3$$

$$\Delta_\phi^2 \vec{Q} := \sum_{j=0}^{3} \Delta_\phi^2 Q_j, \quad \Delta_\phi^2 \vec{P} := \sum_{j=1}^{3} \Delta_\phi^2 P_j$$

$$P_j \phi := \frac{d\phi(e^{it\sigma_{j/2}} g)}{dt}\Big|_{t=0}$$

$$\min_{\phi \in L^2(\mathrm{SU}(2))} \left\{ \Delta_\phi^2 \vec{Q} \,|\, \Delta_\phi^2 \vec{P} \le E \right\}$$

$$= \max_{s>0} 1 - (s(E + 1/4) - s b_2(\frac{8}{s}) / 16)^2$$

Function ϕ realizing the minimum is given by using

$$\mathrm{se}_2(\frac{\theta}{4}, -\frac{8}{s_E})$$

Conclusion

- We have proposed a method with Inverse Fourier transform as a unified approach for estimation of group action
- Using this method, we have derived the optimal estimator with energy constraint in several groups.
- We have shown that entanglement of input and output cannot improve under energy constraint.
- We have applied it to uncertainty relation.

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